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Spontaneous polarization of the Kondo problem associated with the higher-spin six-vertex model

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Abstract. We study the multi-channel Kondo model associated with an integrable higher-spin analogue of the anti-ferroelectric six-vertex model, which is constructed by inserting spin $\frac{1}{2}$ to spin 1 lines: $\cdots\mathbb{C}^3\otimes\mathbb{C}^3\otimes\mathbb{C}^3\otimes\mathbb{C}^2\otimes\mathbb{C}^3\otimes\mathbb{C}^3\otimes\mathbb{C}^3\cdots$. We formulate the problem in terms of representation theory of quantum affine algebra $U_q(\widehat{sl_2})$ [1]. We derive an exact formula for the spontaneous staggered polarization for our model, which corresponds to Baxter's formula [2] for the six-vertex model.

1. Introduction

In 1973 Baxter [2] studied spontaneous staggered polarization of the six-vertex model. He derived an exact formula for this quantity by the transfer matrix method:

$$\frac{(q^2;q^2)_\infty^2}{(-q^2;q^2)_\infty^2}. \tag{1}$$

Here we have used the standard notation

$$(z;p)_\infty=\prod_{n=0}^{\infty}(1-p^nz).$$

In 1976 Baxter [3] invented the corner transfer matrix method. The calculation of the spontaneous staggered polarization was reduced to counting the multiplicities of the eigenvalues of the corner transfer matrix. It was recognized that in many interesting cases the eigenvalue of the corner transfer matrix can be described in terms of the characters of affine Lie algebras. Kyoto–School [1, 4] gave mathematical explanations of the corner transfer matrix method, and at the same time they invented the representation theoretical approach to solvable lattice models. Kyoto–School's approach reproduces Baxter's formula (1) and makes it possible to calculate the quantities which cannot be calculated by the corner transfer matrix method. Kyoto–School's methods have been applied to various problems [5–8]. Nakayashiki [9] introduced new-type vertex operators and gave the mathematical formulation of the usual Kondo model.

In this paper we consider the multi-channel Kondo problem [10] associated with an integrable higher-spin analogue of the anti-ferroelectric six-vertex model, which is constructed by inserting spin $\frac{1}{2}$ to spin 1 lines:

$$\cdots\mathbb{C}^3\otimes\mathbb{C}^3\otimes\mathbb{C}^3\otimes\mathbb{C}^2\otimes\mathbb{C}^3\otimes\mathbb{C}^3\otimes\mathbb{C}^3\cdots.$$

This problem has quantum affine symmetry $U_q(\widehat{sl_2})$. Our main result is an exact formula for spontaneous staggered polarization:

$$-\frac{1}{1-q^4}\frac{(q^{16};q^{16})_\infty}{(q^4;q^4)_\infty}\left\{(1+q^4)\frac{(-q^4;q^8)_\infty}{(-q^4;q^4)^2_\infty}-2q^2\frac{(-q^8;q^{16})^2_\infty}{(-q^2;q^4)^2_\infty}-4q^4\frac{(-q^{16};q^{16})^2_\infty}{(-q^2;q^4)^2_\infty}\right\}.$$

Now, a few words about the organization of the paper. In section 2 we define the problem and state the main result. In section 3 we derive an exact formula for spontaneous staggered polarization.

2. Problem and result

The purpose of this section is to set the problem and summarize the main result.

2.1. Quantum affine algebra $U_q(\widehat{sl_2})$

We follow the notation of [1]. We give definitions of quantum affine Lie algebras $U_q(\widehat{sl_2})$, highest weight modules, and principal evaluation modules.

Consider a free Abelian group on the letters $\Lambda_0, \Lambda_1, \delta$:

$$P = \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda_1 \oplus \mathbb{Z}\delta.$$

Define the simple roots α_0, α_1 and an element ρ by

$$\alpha_0 + \alpha_1 = \delta \qquad \Lambda_1 = \Lambda_0 + \frac{\alpha_1}{2} \qquad \rho = \Lambda_0 + \Lambda_1.$$

Let (h_0, h_1, d) be a basis of $P^* = \mathrm{Hom}\,(P, \mathbb{Z})$ dual to $(\Lambda_0, \Lambda_1, \delta)$. Define a symmetric bilinear form by

$$(\Lambda_0, \Lambda_0) = 0 \qquad (\Lambda_0, \alpha_1) = 0 \qquad (\Lambda_0, \delta) = 1$$
$$(\alpha_1, \alpha_1) = 2 \qquad (\alpha_1, \delta) = 0 \qquad (\delta, \delta) = 0.$$

Regarding $P^* \subset P$, via this bilinear form, we have the identification

$$h_0 = \alpha_0 \qquad h_1 = \alpha_1 \qquad d = \Lambda_0.$$

We use the symbol

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

The quantum affine algebra $U_q(\widehat{sl_2})$ is an algebra with 1 over $\mathbb{C}$, defined on the generators e_0, e_1, f_0, f_1 and q^h $(h \in P^*)$ through the defining relations:

$$q^h q^{h'} = q^{h+h'} \qquad q^0 = 1$$
$$q^h e_i q^{-h} = q^{(\alpha_i, h)} e_i \qquad q^h f_i q^{-h} = q^{-(\alpha_i, h)} f_i$$
$$[e_i, f_j] = \delta_{ij}\frac{t_i - t_i^{-1}}{q - q^{-1}}$$
$$e_i^3 e_j - [3]e_i^2 e_j e_i + [3]e_i e_j e_i^2 - e_j e_i^3 = 0 \qquad (i \neq j)$$
$$f_i^3 f_j - [3]f_i^2 f_j f_i + [3]f_i f_j f_i^2 - f_j f_i^3 = 0 \qquad (i \neq j).$$

Here $t_i = q^{h_i}$. We write $U_q'(\widehat{sl_2})$ for the subalgebra of $U_q(\widehat{sl_2})$ generated by $e_0, e_1, f_0, f_1, q^{h_0}, q^{h_1}$, and $U_q(sl_2)$ by e_1, f_1, q^{h_1}. We define the coproduct Δ by

$$\Delta(q^h) = q^h \otimes q^h \qquad \Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i \qquad \Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i.$$

We define the irreducible highest weight module. Set $P_+ = \mathbb{Z}_{\geqslant 0}\Lambda_0 \oplus \mathbb{Z}_{\geqslant 0}\Lambda_1$. For $\lambda \in P_+$, a $U_q(\widehat{sl_2})$ module $V(\lambda)$ is called an irreducible highest weight module with highest weight λ if the following conditions are satisfied: there exists a nonzero vector $|\lambda\rangle \in V(\lambda)$, called the highest weight vector, such that $q^h|\lambda\rangle = q^{(\lambda,h)}|\lambda\rangle (h \in P^*)$, $e_i|\lambda\rangle = f_i^{(\lambda,h_i)+1}|\lambda\rangle = 0$ $(i = 0, 1)$, and $V(\lambda) = U_q(\widehat{sl_2})|\lambda\rangle$. We say that $V(\lambda)$ has level k if $t_0 t_1|\lambda\rangle = q^k|\lambda\rangle$. When $V(\lambda)$ has level k, the weight $\lambda = (k-m)\Lambda_0 + m\Lambda_1$ $(m = 0, \ldots, k)$. In this paper we use level 2 modules: $V(2\Lambda_0)$, $V(\Lambda_0 + \Lambda_1)$, $V(2\Lambda_1)$.

We define the principal evaluation modules V_ζ of the subalgebra $U_q'(\widehat{sl_2})$. Let V be a module of $U_q(sl_2)$. We equip V_ζ with a $U_q'(\widehat{sl_2})$-module structure by setting

$$\begin{aligned}
&e_0(v_\epsilon \otimes \zeta^m) = (f_1 v_\epsilon) \otimes \zeta^{m+1} \qquad && e_1(v_\epsilon \otimes \zeta^m) = (e_1 v_\epsilon) \otimes \zeta^{m+1}\\
&f_0(v_\epsilon \otimes \zeta^m) = (e_1 v_\epsilon) \otimes \zeta^{m-1} \qquad && f_1(v_\epsilon \otimes \zeta^m) = (f_1 v_\epsilon) \otimes \zeta^{m-1}\\
&t_0 = t_1^{-1} \qquad t_1(v_\epsilon \otimes \zeta^m) = (t_1 v_\epsilon) \otimes \zeta^m. &&
\end{aligned}$$

2.2. *R-matrix and lattice model*

In this section we define our two-dimensional lattice model, and summarize the main result. Let $V_\zeta^{(1)} \simeq \mathbb{C}^3$ and $V_\zeta^{(\frac{1}{2})} \simeq \mathbb{C}^2$ be the $U_q(\widehat{sl_2})$ principal modules. We fix real numbers q and ζ as

$$-1 < q < 0 \qquad 1 < \zeta < (-q)^{-1}$$

in the following. The Boltzmann weights of our model are specified by the spin $(1,1)$ R-matrix intertwiner $R^{(1,1)}(\zeta)$ and the spin $(\frac{1}{2}, 1)$ R-matrix intertwiner $R^{(\frac{1}{2},1)}(\zeta)$. The spin $(1,1)$ R-matrix intertwiner $R^{(1,1)}(\zeta_1/\zeta_2) : V_{\zeta_1}^{(1)} \otimes V_{\zeta_2}^{(1)} \to V_{\zeta_2}^{(1)} \otimes V_{\zeta_1}^{(1)}$ is given by

$$R^{(1,1)}(\zeta) = \frac{1}{\kappa^{(1,1)}(\zeta)} \begin{pmatrix}
a_1 &&&&&&&& \\
& a_2 && a_3 &&&&& \\
&& a_4 && a_5 && a_6 && \\
& a_3 && a_2 &&&&& \\
&& a_5 && a_7 && a_5 && \\
&&&&& a_2 && a_3 & \\
&& a_6 && a_5 && a_4 && \\
&&&&& a_3 && a_2 & \\
&&&&&&&& a_1
\end{pmatrix}. \tag{2}$$

Here we set

$$\kappa^{(1,1)}(\zeta) = \zeta^2 \frac{1 - q^2\zeta^{-2}}{1 - q^2\zeta^2}$$

and

$$\begin{aligned}
&a_1 = 1 \qquad a_2 = (1-\zeta^2)q^2/d_4 \qquad a_3 = (1-q^4)\zeta/d_4\\
&a_4 = (1-\zeta^2)(q^2-\zeta^2)q^2/d_2 d_4 \qquad a_5 = (1-\zeta^2)(1-q^4)q\zeta/d_2 d_4\\
&a_6 = (1-q^2)(1-q^4)\zeta^2/d_2 d_4\\
&a_7 = a_2 + a_6 \qquad d_2 = 1 - q^2\zeta^2 \qquad d_4 = 1 - q^4\zeta^2.
\end{aligned}$$

It is the Boltzmann weight, a_6, that dominates at low temperature, i.e., when q is nearly equal to zero. The R-matrix $R^{(1,1)}(\zeta)$ satisfies unitarity and crossing symmetry:

$$R^{(1,1)}(\zeta)R^{(1,1)}(\zeta^{-1}) = I \qquad R^{(1,1)}(-q^{-1}\zeta)_{k,l}^{k',l'} = R^{(1,1)}(\zeta^{-1})_{2-k',l}^{2-k,l'}.$$

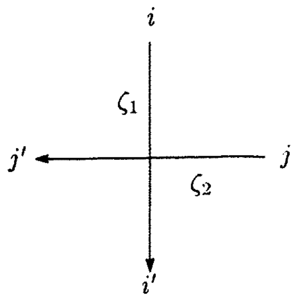

Figure 1. Boltzmann weights $R^{(1,1)}(f)$.

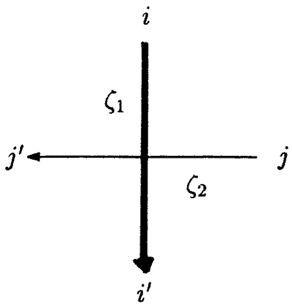

Figure 2. Boltzmann weights $R^{(\frac{1}{2},1)}(f)$.

Let us define the spin $(\frac{1}{2},1)$ R-matrix intertwiner $R^{(\frac{1}{2},1)}(\zeta_1/\zeta_2): V_{\zeta_1}^{(\frac{1}{2})}\otimes V_{\zeta_2}^{(1)}\to V_{\zeta_2}^{(1)}\otimes V_{\zeta_1}^{(\frac{1}{2})}$ by

$$R^{(\frac{1}{2},1)}(\zeta)=\frac{1}{\kappa^{(\frac{1}{2},1)}(\zeta)}\begin{pmatrix} b_1 & & & & & \\ & b_2 & & b_4 & & \\ & & b_3 & & b_4 & \\ & b_4 & & b_3 & & \\ & & b_4 & & b_2 & \\ & & & & & b_1 \end{pmatrix}. \tag{3}$$

Here we set

$$\kappa^{(\frac{1}{2},1)}(\zeta)=\zeta\frac{(q^5\zeta^2;q^4)_\infty(q^3\zeta^{-2};q^4)_\infty}{(q^5\zeta^{-2};q^4)_\infty(q^3\zeta^2;q^4)_\infty}$$

and

$$b_1=1\qquad b_2=\frac{(1-q\zeta^2)q}{1-q^3\zeta^2}\qquad b_3=\frac{(q-\zeta^2)q}{1-q^3\zeta^2}\qquad b_4=\sqrt{1+q^2}\,\frac{(1-q^2)\zeta}{1-q^3\zeta^2}.$$

It is the Boltzmann weight, b_4, that dominates at low temperature, i.e., when q is nearly equal to zero. The R-matrix $R^{(\frac{1}{2},1)}(\zeta)$ satisfies unitarity and crossing symmetry:

$$R^{(\frac{1}{2},1)}(\zeta)R^{(\frac{1}{2},1)}(\zeta^{-1})=I\qquad R^{(\frac{1}{2},1)}(-q^{-1}\zeta)_{k,l}^{k',l'}=R^{(\frac{1}{2},1)}(\zeta^{-1})_{1-k',l}^{1-k,l'}.$$

A lattice vertex associated with the interaction of a spin 1 and spin 1 line has spin variables $i,i'=(0,1,2)$ and $j,j'=(0,1,2)$, and spectral parameters $\zeta_1,\zeta_2\in\mathbb{C}$. A Boltzmann weight $R^{(1,1)}(\zeta_1/\zeta_2)_{i',j'}^{i,j}$ is attached to the configuration of these variables shown in figure 1. A lattice vertex associated with the interaction of a spin $\frac{1}{2}$ and spin 1 line has spin variables $i,i'=(0,1)$ and $j,j'=(0,1,2)$, and spectral parameters $\zeta_1,\zeta_2\in\mathbb{C}$. A Boltzmann weight $R^{(\frac{1}{2},1)}(\zeta_1/\zeta_2)_{i',j'}^{i,j}$ is attached to the configuration of these variables shown in figure 2.

Now we consider the finite lattice in figure 3 under special boundary conditions.

Our model has $2N+1$ vertical lines with spectral parameter ζ and $2N$ horizontal lines with spectral parameter 1, where $N\in\mathbb{N}$. The boundary conditions a_j,b_j,c_j,d_j $(j=1,2,\ldots,2N)$ are fixed in the following four cases, and their ground states are shown in figure 4:

(1) $(\Lambda_0+\Lambda_1,2\Lambda_0)$-case:

$$a_j=1+(-1)^{N+j+1}\qquad b_j=1\qquad c_j=1\qquad d_j=1+(-1)^N$$

(2) $(\Lambda_0+\Lambda_1,2\Lambda_1)$-case:

$$a_j=1+(-1)^{N+j}\qquad b_j=1\qquad c_j=1\qquad d_j=1+(-1)^{N+1}$$

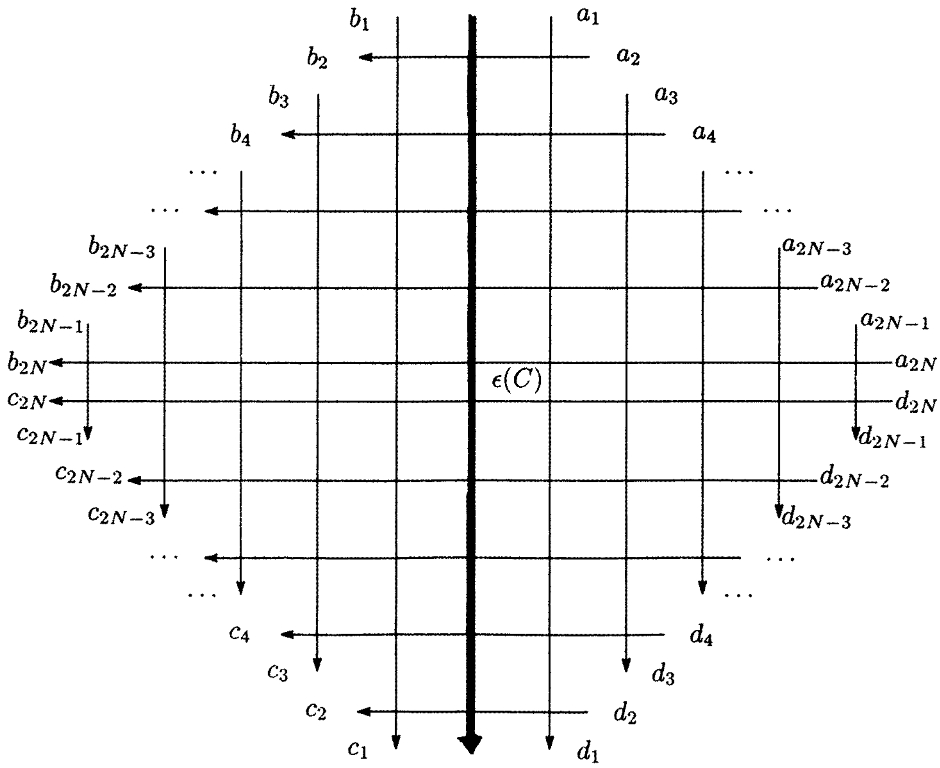

Figure 3. Lattice model.

(3) $(2\Lambda_0, \Lambda_0 + \Lambda_1)$-case:

$$a_j = 1 \qquad b_j = 1 + (-1)^{N+1} \qquad c_j = 1 + (-1)^{N+j} \qquad d_j = 1$$

(4) $(2\Lambda_1, \Lambda_0 + \Lambda_1)$-case:

$$a_j = 1 \qquad b_j = 1 + (-1)^{N} \qquad c_j = 1 + (-1)^{N+j+1} \qquad d_j = 1.$$

Let us set a configuration C to be an assignment of spins. Hence there are $4N^2 + 4N + 1$ configurations for each boundary condition (λ, μ). We introduce a probability measure in the set of all configurations, assigning a statistical weight $W_N^{(\lambda,\mu)}(C)$ to each configuration C attached to the boundary condition (λ, μ). The weight $W_N^{(\lambda,\mu)}(C)$ is given as the product over all vertices

$$W_N^{(\lambda,\mu)}(C) = \prod_{vertex} R^{(1,1)}(\zeta)_{i'j'}^{ij} \prod_{vertex} R^{(\frac{1}{2},1)}(\zeta)_{k'l'}^{kl}.$$

Here we muliply the R-matrices under the boundary condition (λ, μ). The probability for the configuration C to take place is $\frac{1}{Z_N^{(\lambda,\mu)}} W_N^{(\lambda,\mu)}(C)$, where

$$Z_N^{(\lambda,\mu)} = \sum_C W_N^{(\lambda,\mu)}(C).$$

This normalization factor $Z_N^{(\lambda,\mu)}$ is called the partition function. The probability that the vertex of the centre of our lattice takes value ϵ is given as follows:

$$P_\epsilon^{(\lambda,\mu)}(N) = \frac{\sum_{C \ (\text{s.t.}\epsilon(C)=\epsilon)} W_N^{(\lambda,\mu)}(C)}{Z_N^{(\lambda,\mu)}}. \tag{4}$$

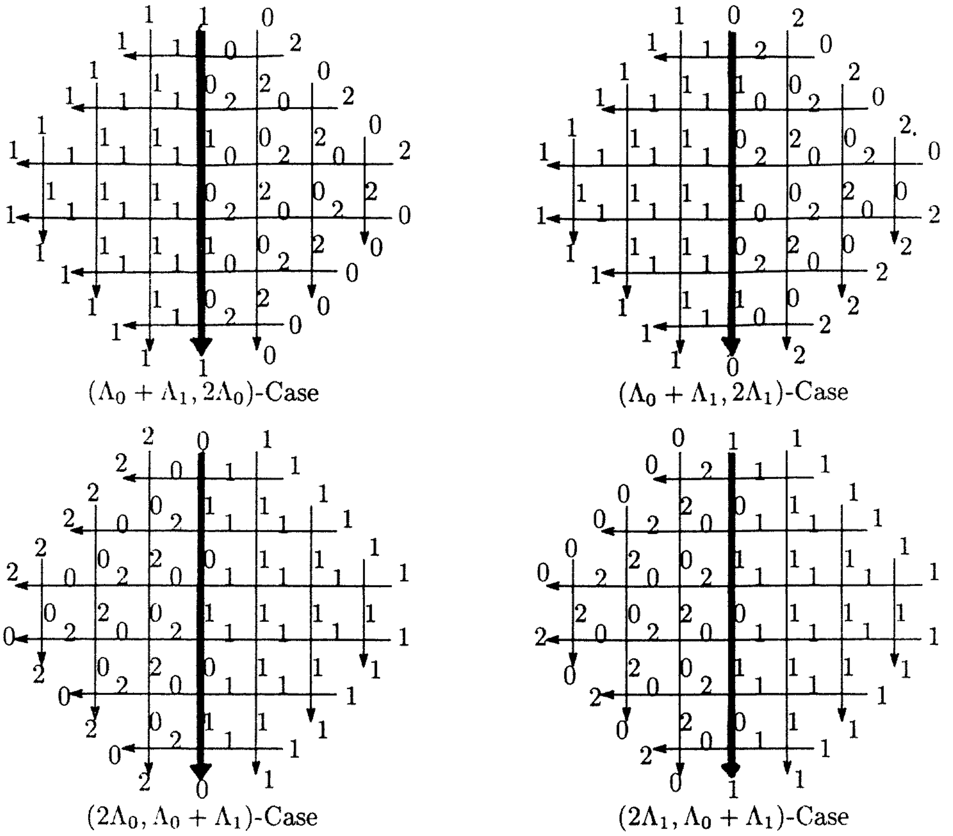

Figure 4. Ground states.

Here the superscript (λ, μ) represents the boundary conditions. In this paper we are interested in the probability functions in the infinite volume limit defined by

$$P_\epsilon^{(\lambda,\mu)} = \lim_{N\to\infty} P_\epsilon^{(\lambda,\mu)}(N). \tag{5}$$

We consider the infinite volume limit in the region given by

$$-1 < q < 0 \qquad 1 < \zeta < (-q)^{-1}.$$

From symmetry arguments, we have the relations between the probability functions:

$$P_\epsilon^{(\Lambda_0+\Lambda_1,2\Lambda_0)} = P_{1-\epsilon}^{(\Lambda_0+\Lambda_1,2\Lambda_1)} = P_\epsilon^{(2\Lambda_1,\Lambda_0+\Lambda_1)} = P_{1-\epsilon}^{(2\Lambda_0,\Lambda_0+\Lambda_1)} \qquad (\epsilon = 0, 1).$$

We show that the probability functions have the following formulae:

$$\begin{aligned} P_0^{(\Lambda_0+\Lambda_1,2\Lambda_0)} = {} & \frac{1}{2}\frac{(q^4;q^2)_\infty}{(q^4;-q^2)_\infty} - \frac{1}{1-q^2}\frac{(q^{16};q^{16})_\infty}{(q^4;q^4)_\infty}\left\{\frac{1}{2}\left(-1-q^2+\frac{4q^4}{1-q^4}\right)\frac{(-q^8;q^{16})_\infty^2}{(-q^2;q^4)_\infty^2}\right. \\ & \left. + \left(-q^2-q^4+\frac{4q^6}{1-q^4}\right)\frac{(-q^{16};q^{16})_\infty^2}{(-q^2;q^4)_\infty^2} + \left(1+q^2-\frac{2q^2}{1-q^4}\right)\frac{(-q^4;q^8)_\infty}{(-q^4;q^4)_\infty^2}\right\} \end{aligned} \tag{6}$$

and

$$P_1^{(\Lambda_0+\Lambda_1,2\Lambda_0)} = \frac{1}{2}\frac{(q^4;q^2)_\infty}{(q^4;-q^2)_\infty} - \frac{1}{1-q^2}\frac{(q^{16};q^{16})_\infty}{(q^4;q^4)_\infty}\left\{\frac{1}{2}\left(-1-q^2+\frac{4q^2}{1-q^4}\right)\frac{(-q^8;q^{16})_\infty^2}{(-q^2;q^4)_\infty^2}\right.$$

$$+\left(-q^2-q^4+\frac{4q^4}{1-q^4}\right)\frac{(-q^{16};q^{16})_\infty^2}{(-q^2;q^4)_\infty^2}-\frac{2q^4}{1-q^4}\frac{(-q^4;q^8)_\infty}{(-q^4;q^4)_\infty^2}\Bigg\}. \tag{7}$$

The following is the main result of this paper, which is a direct consequence of relations (6) and (7).

Main result 2.1. *The spontaneous staggered polarization of our model has the following infinite product formula:*

$$P_0^{(\Lambda_0+\Lambda_1,2\Lambda_0)}-P_1^{(\Lambda_0+\Lambda_1,2\Lambda_0)}=-\frac{1}{1-q^4}\frac{(q^{16};q^{16})_\infty}{(q^4;q^4)_\infty}$$
$$\times\left\{(1+q^4)\frac{(-q^4;q^8)_\infty}{(-q^4;q^4)_\infty^2}-2q^2\frac{(-q^8;q^{16})_\infty^2}{(-q^2;q^4)_\infty^2}-4q^4\frac{(-q^{16};q^{16})_\infty^2}{(-q^2;q^4)_\infty^2}\right\}. \tag{8}$$

In fact, the spontaneous staggered polarization is independent of the spectral parameter ζ.

Remark. From relation (9) and the trace formula (10), we get

$$P_0^{(\Lambda_0+\Lambda_1,2\Lambda_0)}+P_1^{(\Lambda_0+\Lambda_1,2\Lambda_0)}=1.$$

In what follows we explain how to derive this formula.

3. Derivation

The purpose of this section is to show the main result.

3.1. Infinite volume limit

We consider the infinite volume limit $N\to\infty$. For simplicity, we concentrate on the boundary condition $(\Lambda_0+\Lambda_1,2\Lambda_0)$.

A path is defined as a sequence of $0,1,2$, denoted by $|p\rangle=\{p(j)\}_{j\geqslant 1}$. For weights $\lambda=2\Lambda_0,\Lambda_0+\Lambda_1,2\Lambda_1$, consider the set of paths $P_{2\Lambda_0},P_{\Lambda_0+\Lambda_1},P_{2\Lambda_1}$ by

$$P_{2\Lambda_0}=\{|p\rangle|p(j)=1+(-1)^j,\ \text{for } j\gg 0\}$$
$$P_{\Lambda_0+\Lambda_1}=\{|p\rangle|p(j)=1,\ \text{for } j\gg 0\}$$
$$P_{2\Lambda_1}=\{|p\rangle|p(j)=1+(-1)^{j+1},\ \text{for } j\gg 0\}.$$

The infinite lattice, so defined, may be split into six pieces, consisting of four corners and two half columns (see figure 5). The associated corner transfer matrices are labelled $A(\zeta),B(\zeta),C(\zeta)$ and $D(\zeta)$. Two lines are labelled $\Phi_{UP,\epsilon}(\zeta)$ and $\Phi_{LOW,\epsilon}(\zeta)$.

Following Baxter we define the corner transfer matrices $O^{(1)}(\zeta),O^{(2)}(\zeta)$ in the infinite volume limit $N\to\infty$, by the sum over the spin configurations in the interior,

$$(O^{(b)}(\zeta))_{|p\rangle}^{|p'\rangle}=\sum_{\text{interior edges}}\prod R_{\epsilon_1\epsilon_2}^{\epsilon_1'\epsilon_2'}$$

where we take summations with the following boundary conditions related to the superscripts $b=1,2$. For $b=1$, the paths $|p\rangle,|p'\rangle$ belong to the set of paths $P_{\Lambda_0+\Lambda_1}$, and the north-west boundary is fixed by $b=1$ (see figure 6). For $b=2$, the paths $|p\rangle,|p'\rangle$ belong to the set of paths $P_{2\Lambda_0}$, and the north-west boundary is fixed by $b=2$ (see figure 7). The corner transfer matrices $O^{(1)}(\zeta)$ and $O^{(2)}(\zeta)$ act on the path spaces $P_{\Lambda_0+\Lambda_1}$ and $P_{2\Lambda_0}$, respectively.

Define the operators $S:P_{2\Lambda_0}\cup P_{\Lambda_0+\Lambda_1}\cup P_{2\Lambda_1}\to P_{2\Lambda_0}\cup P_{\Lambda_0+\Lambda_1}\cup P_{2\Lambda_1}$ by $p(j)\to 2-p(j)$. The corner transfer matrices $A(\zeta),D(\zeta)$ act on the path space $P_{2\Lambda_0}$. The corner transfer

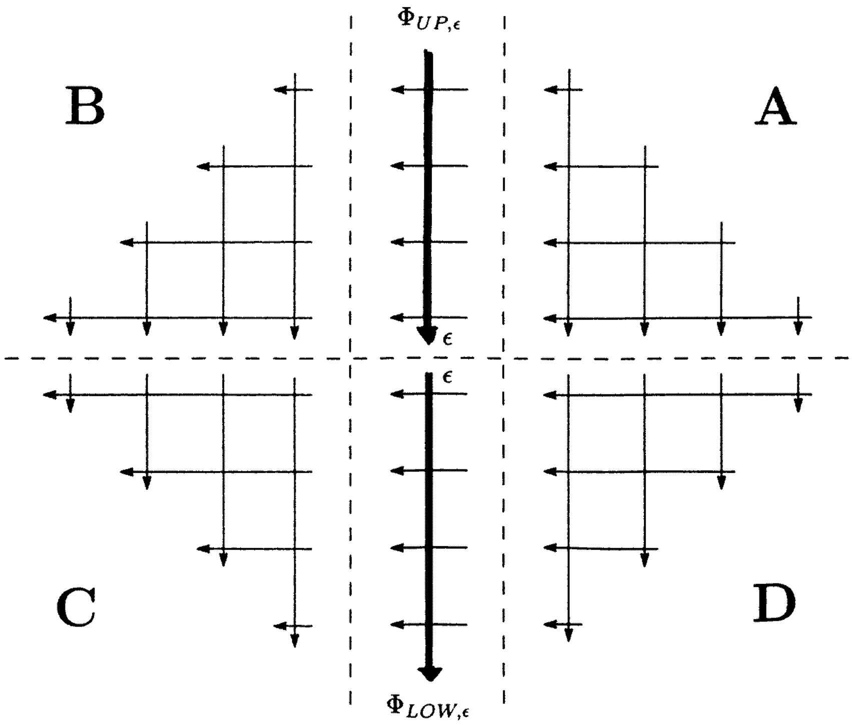

Figure 5. Subdivision of the lattice into quadrants.

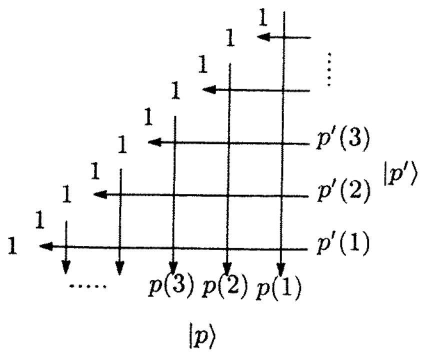

Figure 6. Corner transfer matrix $O^{(1)}_{(\zeta)}$.

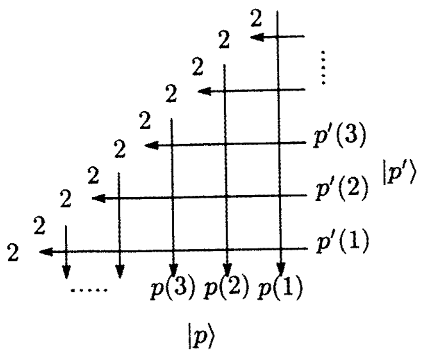

Figure 7. Corner transfer matrix $O^{(2)}_{(\zeta)}$.

matrices $B(\zeta), C(\zeta)$ act on the path space $P_{\Lambda_0+\Lambda_1}$. Using the crossing symmerty of the R-matrix, we can write

$$A(\zeta) = O^{(2)}(-q^{-1}\zeta^{-1}) \cdot S|_{P_{2\Lambda_0}} \qquad B(\zeta) = O^{(1)}(\zeta)|_{P_{\Lambda_0+\Lambda_1}}$$
$$C(\zeta) = S \cdot O^{(1)}(-q^{-1}\zeta^{-1})|_{P_{\Lambda_0+\Lambda_1}} \qquad D(\zeta) = S \cdot O^{(2)}(\zeta) \cdot S|_{P_{2\Lambda_0}}.$$

Baxter's argument [11] implies that $O^{(1)}(\zeta) = \text{const.}\zeta^{-H_{CTM}|_{P_{\Lambda_0+\Lambda_1}}}$, and $O^{(2)}(\zeta) = \text{const.}\zeta^{-H_{CTM}|_{P_{2\Lambda_0}}}$, where $H_{CTM}|_{P_\lambda}$ does not depend on the spectral parameter ζ. Kyoto–School's conjecture is to identify the path spaces $P_{2\Lambda_0}$, $P_{\Lambda_0+\Lambda_1}$ and $P_{2\Lambda_1}$ with the highest weight modules of $U_q(\widehat{sl_2})$, $V(2\Lambda_0)$, $V(\Lambda_0+\Lambda_1)$ and $V(2\Lambda_1)$, which has been proved at $q=0$ by a crystal base argument. Under this identification the degree operator $H_{CTM}|_{P_\lambda}$ is realized as $H_{CTM}|_{P_\lambda} = D|_{V(\lambda)} = -\rho + (\rho, \lambda)$, where $\lambda = 2\Lambda_0, \Lambda_0+\Lambda_1$ and $2\Lambda_1$. The semi-infinite chain $\Phi_{UP,\epsilon}(\zeta)$ is identified with the type-I vertex operator $\Phi_{2\Lambda_0,\epsilon}^{\Lambda_0+\Lambda_1}(\zeta)$ defined by

$$\Phi_{2\Lambda_0}^{\Lambda_0+\Lambda_1}(\zeta) = \sum_{\epsilon=0,1} \Phi_{2\Lambda_0,\epsilon}^{\Lambda_0+\Lambda_1}(\zeta) \otimes v_\epsilon$$

where the $U_q(\widehat{sl_2})$-intertwiner $\Phi_{2\Lambda_0}^{\Lambda_0+\Lambda_1}(\zeta)$ is defined by

$$\Phi_{2\Lambda_0}^{\Lambda_0+\Lambda_1}(\zeta) : V(2\Lambda_0) \longrightarrow V(\Lambda_0+\Lambda_1) \otimes V_\zeta^{(\frac{1}{2})}.$$

The semi-infinite chain $\Phi_{LOW,\epsilon}(\zeta)$ is identified with the type-I vertex operator

$$\Phi_{LOW,\epsilon}(\zeta) = S \cdot \Phi_{\Lambda_0+\Lambda_1,1-\epsilon}^{2\Lambda_0}(\zeta) \cdot S.$$

The type-I vertex operator $\Phi_{\Lambda_0+\Lambda_1,\epsilon}^{2\Lambda_0}(\zeta)$ is defined in the same manner

$$\Phi_{\Lambda_0+\Lambda_1}^{2\Lambda_0}(\zeta) = \sum_{\epsilon=0,1} \Phi_{\Lambda_0+\Lambda_1,\epsilon}^{2\Lambda_0}(\zeta) \otimes v_\epsilon$$

where the $U_q(\widehat{sl_2})$-intertwiner $\Phi_{\Lambda_0+\Lambda_1}^{2\Lambda_0}(\zeta)$ is defined by

$$\Phi_{\Lambda_0+\Lambda_1}^{2\Lambda_0}(\zeta) : V(\Lambda_0+\Lambda_1) \longrightarrow V(2\Lambda_0) \otimes V_\zeta^{(\frac{1}{2})}.$$

We assume, along the line of the XXZ-chain [1], that the vertex operators satisfy the homogeneity condition,

$$\xi^{-D} \cdot \Phi_{\Lambda_0+\Lambda_1,\epsilon}^{2\Lambda_0}(\zeta) \cdot \xi^D = \Phi_{\Lambda_0+\Lambda_1,\epsilon}^{2\Lambda_0}(\zeta/\xi).$$

From this condition we have

$$P_\epsilon^{(\Lambda_0+\Lambda_1,2\Lambda_0)} = \frac{\mathrm{tr}_{V(2\Lambda_0)}(q^{2D}\Phi_{\Lambda_0+\Lambda_1,1-\epsilon}^{2\Lambda_0}(-q^{-1}\zeta)\Phi_{2\Lambda_0,\epsilon}^{\Lambda_0+\Lambda_1}(\zeta))}{\sum_{\epsilon=0,1}\mathrm{tr}_{V(2\Lambda_0)}(q^{2D}\Phi_{\Lambda_0+\Lambda_1,1-\epsilon}^{2\Lambda_0}(-q^{-1}\zeta)\Phi_{2\Lambda_0,\epsilon}^{\Lambda_0+\Lambda_1}(\zeta))}.$$

We adopt the normalizations

$$\langle \Lambda_0+\Lambda_1|\Phi_{2\Lambda_0,1}^{\Lambda_0+\Lambda_1}(\zeta)|2\Lambda_0\rangle = 1 \qquad \langle 2\Lambda_0|\Phi_{\Lambda_0+\Lambda_1,0}^{2\Lambda_0}(\zeta)|\Lambda_0+\Lambda_1\rangle = 1.$$

The vertex operator $\Phi_{\Lambda_0+\Lambda_1,\epsilon}^{2\Lambda_0}(-q^{-1}\zeta)$ is identified with the dual-vertex operator

$$\Phi_{\Lambda_0+\Lambda_1,\epsilon}^{2\Lambda_0}(-q^{-1}\zeta) = \Phi_{\Lambda_0+\Lambda_1,1-\epsilon}^{2\Lambda_0\,*}(\zeta).$$

The dual-vertex operator $\Phi_{\Lambda_0+\Lambda_1,\epsilon}^{2\Lambda_0\,*}(\zeta)$ is defined by

$$\Phi_{\Lambda_0+\Lambda_1,\epsilon}^{2\Lambda_0\,*}(\zeta)|v\rangle = \Phi_{\Lambda_0+\Lambda_1}^{2\Lambda_0\,*}(\zeta)(|v\rangle \otimes v_\epsilon)$$

where the $U_q(\widehat{sl_2})$-intertwiner $\Phi_{\Lambda_0+\Lambda_1}^{2\Lambda_0\,*}(\zeta)$ is defined by

$$\Phi_{\Lambda_0+\Lambda_1}^{2\Lambda_0\,*}(\zeta) : V(\Lambda_0+\Lambda_1) \otimes V_\zeta^{(\frac{1}{2})} \longrightarrow V(2\Lambda_0).$$

We adopt the normalization

$$\langle 2\Lambda_0|\Phi_{\Lambda_0+\Lambda_1,1}^{2\Lambda_0\,*}(\zeta)|\Lambda_0+\Lambda_1\rangle = 1.$$

Because the operator $\sum_{\epsilon=0,1}\Phi_{\Lambda_0+\Lambda_1,\epsilon}^{2\Lambda_0\,*}(\zeta)\Phi_{2\Lambda_0,\epsilon}^{\Lambda_0+\Lambda_1}(\zeta)$ commutes with $U_q(\widehat{sl_2})$ on the irreducible module $V(2\Lambda_0)$, it becomes a constant $g_{2\Lambda_0}^{-1}$ on $V(2\Lambda_0)$. The constant $g_{2\Lambda_0}^{-1}$ can be determined by solving the q-KZ equation for variables ζ_1/ζ_2, which is satisfied by the vacuum expectation value $\langle 2\Lambda_0|\Phi_{\Lambda_0+\Lambda_1,\epsilon}^{2\Lambda_0\,*}(\zeta_1)\Phi_{2\Lambda_0,\epsilon}^{\Lambda_0+\Lambda_1}(\zeta_2)|2\Lambda_0\rangle$ [12]:

$$g_{2\Lambda_0}\sum_{\epsilon=0,1}\Phi_{\Lambda_0+\Lambda_1,\epsilon}^{2\Lambda_0\,*}(\zeta)\Phi_{2\Lambda_0,\epsilon}^{\Lambda_0+\Lambda_1}(\zeta)=\mathrm{i}d. \tag{9}$$

We get the following formula:

$$P_\epsilon^{(\Lambda_0+\Lambda_1,2\Lambda_0)}=\frac{\mathrm{tr}_{V(2\Lambda_0)}(q^{2D}\Phi_{\Lambda_0+\Lambda_1,\epsilon}^{2\Lambda_0\,*}(\zeta)\Phi_{2\Lambda_0,\epsilon}^{\Lambda_0+\Lambda_1}(\zeta))}{g_{2\Lambda_0}^{-1}\,\mathrm{tr}_{V(2\Lambda_0)}(q^{2D})}. \tag{10}$$

Here we use

$$g_{2\Lambda_0}^{-1}=(1+q^2)\frac{(q^{12};q^8)_\infty(q^{10};q^4,q^4)_\infty}{(q^8;q^8)_\infty(q^{12};q^4,q^8)_\infty^2}$$

and

$$\mathrm{tr}_{V(2\Lambda_0)}(q^{2D})=(-q^2;q^2)_\infty(-q^4;q^4)_\infty.$$

Here we use the notation

$$(z;p_1,p_2,\ldots,p_k)_\infty=\prod_{m_1,m_2,\ldots,m_k=0}^{\infty}(1-p_1^{m_1}p_2^{m_2}\ldots p_k^{m_k}z).$$

By the same arguments we have the following formulae for the boundary conditions $(\mu,\lambda)=(\Lambda_0+\Lambda_1,2\Lambda_1)$, $(2\Lambda_0,\Lambda_0+\Lambda_1)$ or $(2\Lambda_1,\Lambda_0+\Lambda_1)$:

$$P_\epsilon^{(\mu,\lambda)}=\frac{\mathrm{tr}_{V(\lambda)}(q^{2D}\Phi_{\mu,\epsilon}^{\lambda\,*}(\zeta)\Phi_{\lambda,\epsilon}^{\mu}(\zeta))}{g_\lambda^{-1}\,\mathrm{tr}_{V(\lambda)}(q^{2D})}. \tag{11}$$

Here we use

$$g_{2\Lambda_1}^{-1}=(1+q^2)\frac{(q^{12};q^8)_\infty(q^{10};q^4,q^4)_\infty}{(q^8;q^8)_\infty(q^{12};q^4,q^8)_\infty^2}$$
$$g_{\Lambda_0+\Lambda_1}^{-1}=\frac{(q^6;q^4)_\infty(q^{10};q^4,q^4)_\infty}{(q^4;q^4)_\infty(q^{12};q^4,q^8)_\infty^2}$$

and

$$\mathrm{tr}_{V(2\Lambda_1)}(q^{2D})=(-q^2;q^2)_\infty(-q^4;q^4)_\infty$$
$$\mathrm{tr}_{V(\Lambda_0+\Lambda_1)}(q^{2D})=(-q^2;q^2)_\infty(-q^2;q^4)_\infty.$$

The vertex operators are defined in the same manner. From the cyclic property of the trace, we obtain the following relations:

$$P_\epsilon^{(2\Lambda_0,\Lambda_0+\Lambda_1)}=P_{-\epsilon}^{(\Lambda_0+\Lambda_1,2\Lambda_0)}\qquad P_\epsilon^{(2\Lambda_1,\Lambda_0+\Lambda_1)}=P_{-\epsilon}^{(\Lambda_0+\Lambda_1,2\Lambda_1)}.$$

From symmetries we easily know the following relations:

$$P_\epsilon^{(2\Lambda_0,\Lambda_0+\Lambda_1)}=P_{-\epsilon}^{(2\Lambda_1,\Lambda_0+\Lambda_1)}\qquad P_\epsilon^{(\Lambda_0+\Lambda_1,2\Lambda_0)}=P_{-\epsilon}^{(\Lambda_0+\Lambda_1,2\Lambda_1)}.$$

From the commutation relation of vertex operators [12] and the cyclic property of the trace, we can write down the q-difference equation for parameter ζ_1/ζ_2, which the trace $\mathrm{tr}_{V(2\Lambda_0)}(q^{2D}\Phi_{\Lambda_0+\Lambda_1,\epsilon}^{2\Lambda_0\,*}(\zeta_1)\Phi_{2\Lambda_0,\epsilon}^{\Lambda_0+\Lambda_1}(\zeta_2))$ satisfies. However, we cannot solve this q-difference equation, now. In order to get the exact formulae of the probability functions, we use another method—free field realizations.

3.2. Free field realization

In order to calculate the trace of vertex operators

$$\mathrm{tr}_{V(2\Lambda_0)}(q^{2D}\Phi^{2\Lambda_0\ *}_{\Lambda_0+\Lambda_1,\epsilon}(\zeta)\Phi^{\Lambda_0+\Lambda_1}_{2\Lambda_0,\epsilon}(\zeta))$$

we use the free field realization obtained by Hara [13]. For the readers' convenience, we summarize his result. The formulae in this paper are slightly different from Hara's paper, because his paper includes a few mistakes, which are serious for our needs. We use current-type generators of $U'_q(\widehat{sl_2})$ introduced by Drinfeld. Let A be an algebra generated by $x^{\pm}_m (m \in \mathbb{Z})$, $a_m(m \in \mathbb{Z}_{\neq 0})$, γ and K with relations

$$\begin{aligned}
&\gamma : \text{central}\\
&[a_m, a_n] = \delta_{m+n,0}\frac{[2m]}{m}\frac{\gamma^m - \gamma^{-m}}{q-q^{-1}}\\
&[a_m, K] = 0\\
&Kx^{\pm}_m K^{-1} = q^{\pm 2}x^{\pm}_m\\
&[a_m, x^{\pm}_n] = \pm\frac{[2m]}{[m]}\gamma^{\mp\frac{|m|}{2}}x^{\pm}_{m+n}\\
&x^{\pm}_{m+1}x^{\pm}_n - q^{\pm 2}x^{\pm}_n x^{\pm}_{m+1} = q^{\pm 2}x^{\pm}_m x^{\pm}_{n+1} - x^{\pm}_{n+1}x^{\pm}_m\\
&[x^+_m, x^-_n] = \frac{1}{q-q^{-1}}(\gamma^{\frac{1}{2}(m-n)}\psi_{m+n} - \gamma^{-\frac{1}{2}(m-n)}\varphi_{m+n})
\end{aligned}$$

where

$$\begin{aligned}
\sum_{m=0}^{\infty}\psi_m z^{-m} &= K\exp\left[(q-q^{-1})\sum_{m=1}^{\infty}a_m z^{-m}\right]\\
\sum_{m=0}^{\infty}\varphi_{-m} z^{m} &= K^{-1}\exp\left[-(q-q^{-1})\sum_{m=1}^{\infty}a_{-m} z^{m}\right]
\end{aligned}$$

and $\psi_{-m} = \varphi_m = 0$ for $m > 0$. Drinfeld showed that the algebra A is isomorphic to $U'_q(\widehat{sl_2})$. The Chevalley generators are given by the identification

$$t_0 = \gamma K^{-1} \qquad t_1 = K \qquad e_1 = x^+_0 \qquad f_1 = x^-_0 \qquad e_0 = x^-_1 K^{-1} \qquad f_0 = Kx^+_{-1}.$$

We give explicit constructions of level 2 irreducible highest weight modules. Let us put $\gamma = q^2$ since we want to construct level 2 modules. In what follows we use the parameter $x = -q$ for our convenience. Commutation and anti-commutation relations of bosons and fermions are given by

$$\begin{aligned}
&[a_m, a_n] = \delta_{m+n,0}\frac{[2m]^2}{m}\\
&\{\phi_m, \phi_n\} = \delta_{m+n,0}\eta_m\\
&\eta_m = x^{2m} + x^{-2m}
\end{aligned}$$

with $m, n \in \mathbb{Z} + \frac{1}{2}$ or $\in \mathbb{Z}$ for the Neuveu–Schwartz or Ramond sector, respectively. Fock spaces and vacuum vectors are denoted as F^a, $F^{\phi^{NS}}$, F^{ϕ^R} and $|\mathrm{vac}\rangle$, $|NS\rangle$, $|R\rangle$ for boson, Neuveu–Schwartz and Ramond fermion, respectively. Fermion currents are defined as

$$\phi^{NS}(z) = \sum_{n\in\mathbb{Z}+\frac{1}{2}}\phi^{NS}_n z^{-n} \qquad \phi^R(z) = \sum_{n\in\mathbb{Z}}\phi^R_n z^{-n}.$$

Let us set the degree of the monomial of fermions, $\phi^{NS}_{n_1}\phi^{NS}_{n_2}\ldots\phi^{NS}_{n_s}$ as $n_1+n_2+\cdots+n_s$, and $\phi^R_{n_1}\phi^R_{n_2}\ldots\phi^R_{n_r}$ as $n_1+n_2+\cdots+n_r$. $Q=\mathbb{Z}\alpha$ is the root lattice of sl_2 and $F[Q]$ is the group algebra. We use ∂ as

$$[\partial,\alpha]=2.$$

The irreducible highest weight module $V(2\Lambda_0)$ is identified with the Fock space

$$F^{(0)}_+=F^a\otimes\{(F^{\phi^{NS}}_+\otimes F[2Q])\oplus(F^{\phi^{NS}}_-\otimes \mathrm{e}^\alpha F[2Q])\}$$

where $F^{\phi^{NS}}_+$ represents the subspace of fermion Fock space which is spanned by *even* degree fermions, and $F^{\phi^{NS}}_-$ by *odd* ones. The highest weight vector is $|\mathrm{vac}\rangle\otimes|NS\rangle\otimes 1$. The irreducible highest weight module $V(2\Lambda_1)$ is identified with the Fock space

$$F^{(0)}_-=F^a\otimes\{(F^{\phi^{NS}}_+\otimes \mathrm{e}^\alpha F[2Q])\oplus(F^{\phi^{NS}}_-\otimes F[2Q])\}.$$

The highest weight vector is $|\mathrm{vac}\rangle\otimes|NS\rangle\otimes\mathrm{e}^\alpha$. We define the actions of the Drinfeld generators as

$$\begin{aligned}&\gamma=q^2\qquad K=q^\partial\\&x^\pm(z)=\sum_{m\in\mathbb{Z}}x^\pm_m z^{-m}=E^\pm_<(z)E^\pm_>(z)\phi^{NS}(z)\mathrm{e}^{\pm\alpha}z^{\frac12\pm\frac12\partial}\end{aligned}\tag{12}$$

where

$$E^\pm_<(z)=\exp\left(\pm\sum_{m>0}\frac{a_{-m}}{[2m]}q^{\mp m}z^m\right)\qquad E^\pm_>(z)=\exp\left(\mp\sum_{m>0}\frac{a_m}{[2m]}q^{\mp m}z^{-m}\right).$$

The irreducible highest weight module $V(\Lambda_0+\Lambda_1)$ is identified with the Fock space

$$F^{(1)}=F^a\otimes F^{\phi^R}\otimes\mathrm{e}^{\frac{\alpha}{2}}F[Q]$$

where

$$\phi^R_0|R\rangle=|R\rangle.$$

The highest weight vector is $|\mathrm{vac}\rangle\otimes|R\rangle\otimes\mathrm{e}^{\frac{\alpha}{2}}$. For the actions of the Drinfeld generators, we just replace $\phi^{NS}(z)$ with $\phi^R(z)$ in (12). The free field realizations of vertex operators [13] are constructed by

$$\begin{aligned}&\Phi^{\Lambda_0+\Lambda_1}_{2\Lambda_0,\epsilon}(\zeta)=\zeta^{1-\epsilon}\Phi_\epsilon(\zeta)\\&\Phi^{2\Lambda_0}_{\Lambda_0+\Lambda_1,\epsilon}(\zeta)=-x\zeta^{\frac12-\epsilon}\Phi_\epsilon(\zeta).\end{aligned}$$

Here we set

$$\Phi_1(\zeta)=B_{I,<}(\zeta)B_{I,>}(\zeta)\Omega^R_{NS}(\zeta)\mathrm{e}^{\frac{\alpha}{2}}x^\partial\zeta^{\frac{\partial}{2}}\tag{13}$$

$$\begin{aligned}\Phi_0(\zeta)=\oint\frac{\mathrm{d}w}{2\pi\mathrm{i}}&B_{I,<}(\zeta)E^-_<(w)B_{I,>}(\zeta)E^-_>(w)\Omega^R_{NS}(\zeta)\phi^{NS}(w)\mathrm{e}^{-\frac{\alpha}{2}}x^\partial\zeta^{\frac{\partial}{2}}w^{-\frac{\partial}{2}}\\&\times x^{-2}\zeta^{-1}w^{-\frac32}\frac{(-\frac{w}{x^3\zeta^2};x^4)_\infty}{(-\frac{w}{x\zeta^2};x^4)_\infty}\left\{\frac{w}{1+\frac{w}{x^3\zeta^2}}+\frac{x^5\zeta^2}{1+\frac{x^5\zeta^2}{w}}\right\}\end{aligned}\tag{14}$$

where

$$B_{I,<}(\zeta)=\exp\left(\sum_{n=1}^\infty\frac{[n]a_{-n}}{[2n]^2}(-x^5\zeta^2)^n\right)$$

$$B_{I,>}(\zeta)=\exp\left(-\sum_{n=1}^\infty\frac{[n]a_n}{[2n]^2}(-x^3\zeta^2)^{-n}\right).$$

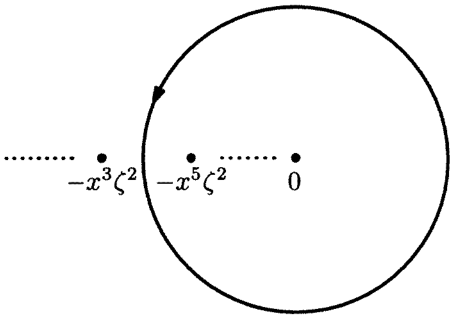

Figure 8. Contor.

Note that the sign of the second term of (14) differs from the one in [13]. The integrand of $\Phi_0(\zeta)$ has poles only at $w = -x^5\zeta^2, -x^3\zeta^2$ except for $w = 0, \infty$ and the contor of integration encloses $w = 0, -x^5\zeta^2$ as in figure 8.

For those of $\Phi^{2\Lambda_0}_{\Lambda_0+\Lambda_1,\epsilon}(\zeta)$ we just replace $\Omega^R_{NS}(\zeta), \phi^{NS}(w)$ with $\Omega^{NS}_R(\zeta), \phi^R(w)$ in (13), (14). The fermion part $\Omega(\zeta)$ are intertwiners between different fermion sectors and satisfy

$$\phi^{NS}(w)\Omega^{NS}_R(\zeta) = x^2\zeta w^{-\frac{1}{2}} \frac{(-\frac{w}{x^3\zeta^2}; x^4)_\infty(-\frac{x^7\zeta^2}{w}; x^4)_\infty}{(-\frac{w}{x\zeta^2}; x^4)_\infty(-\frac{x^5\zeta^2}{w}; x^4)_\infty} \Omega^{NS}_R(\zeta)\phi^R(w)$$

and exactly the same equation, except subscripts for fermion sectors, are exchanged. The homogeneity condition of the fermion parts is given in [13]:

$$\xi^{d^R} \cdot \Omega^R_{NS}(\zeta) \cdot \xi^{-d^{NS}} = \Omega^R_{NS}(\zeta/\xi). \tag{15}$$

The fermion parts $\Omega^R_{NS}(\zeta), \Omega^{NS}_R(\zeta)$ are identified with type-I vertex operators of the two-dimensional Ising model $\Phi^R_{NS}(\zeta), \Phi^{NS}_R(\zeta)$ which were investigated in details [14]:

$$\Omega^R_{NS}(\zeta) = \Phi^R_{NS}\left(-\frac{\mathrm{i}}{x^{\frac{3}{2}}\zeta}\right) \qquad \Omega^{NS}_R(\zeta) = \Phi^{NS}_R\left(-\frac{\mathrm{i}}{x^{\frac{3}{2}}\zeta}\right).$$

For the readers' convenience we summarize the definitions and properties of vertex operators $\Phi^R_{NS}(\zeta)$ and $\Phi^{NS}_R(\zeta)$, which are used later. The type-I vertex operators of the two-dimensional Ising model are operators on Fock spaces

$$\Phi^R_{NS}(\zeta) : F^{\phi^{NS}} \to F^{\phi^R}$$
$$\Phi^{NS}_R(\zeta) : F^{\phi^R} \to F^{\phi^{NS}}.$$

Define the subsectors as

$$\Phi^R_{NS,\sigma}(\zeta) = \Phi^R_{NS}(\zeta)|_{V^{\phi^{NS}}_\sigma} \qquad \Phi^{NS}_{R,\sigma}(\zeta) = P^\sigma\Phi^{NS}_R(\zeta) \qquad \text{for} \quad \sigma = \pm$$

where P^σ denotes the projection onto subspace $F^{\phi^{NS}}_\sigma$. The intertwining relations are given by

$$\phi^{NS}(w)\Phi^{NS,\sigma}_R(\zeta) = f(w\zeta^2)\Phi^{NS,-\sigma}_R(\zeta)\phi^R(w)$$
$$\phi^R(w)\Phi^R_{NS,\sigma}(\zeta) = f(w\zeta^2)\Phi^R_{NS,-\sigma}(\zeta)\phi^{NS}(w).$$

Here we set

$$f(z) = -\sqrt{\frac{2\pi x}{I}}(x^4; x^4)_\infty(-x^4; x^4)^2_\infty \mathrm{sn}(v)$$

where $z = \exp(\pi \mathrm{i} v/I)$ and $\mathrm{sn}(v)$ is the Jacobi elliptic function with half periods $I, \mathrm{i}I'$. Because of the intertwining relations, the following relations hold:

$$\sum_\sigma \Phi^R_{NS,\sigma}(x\zeta)\Phi^{NS,\sigma}_R(\zeta) = g^R \times id_{F^{\phi R}} \tag{16}$$

$$\Phi^{NS,\sigma}_R(x\zeta)\Phi^R_{NS,\sigma}(\zeta) = g^{NS} \times id_{F^{\phi NS}_\sigma} \tag{17}$$

where the constants are

$$g^R = \frac{(x^4; x^4, x^8)^2_\infty}{(x^2; x^4, x^4)_\infty} \qquad g^{NS} = \frac{(x^8; x^4, x^8)^2_\infty}{(x^6; x^4, x^4)_\infty}.$$

We use the following intertwining relations in the next section:

$$\sigma\Phi^{NS,\sigma}_R(\zeta) = \Phi^{NS,\sigma}_R(\zeta)\psi^R_1(\zeta) \tag{18}$$

$$\sigma\Phi^R_{NS,\sigma}(\zeta) = -\mathrm{i}\Phi^R_{NS,-\sigma}(\zeta)\psi^{NS}_1(\zeta) \tag{19}$$

where we set

$$\psi^R_1(\zeta) = \oint \frac{\mathrm{d}z}{2\pi \mathrm{i} z} f^R_0(z)\phi^R(z/\zeta^2)$$

$$\psi^{NS}_1(\zeta) = \oint \frac{\mathrm{d}z}{2\pi \mathrm{i} z} f^{NS}_0(z)\phi^{NS}(z/\zeta^2).$$

Here we set

$$f^{NS}_0(z) = 2\sqrt{x}(x^4; x^4)_\infty(-x^4; x^4)^2_\infty \mathrm{cn}(v) \qquad f^R_0(z) = \sqrt{\frac{2I}{\pi}}\mathrm{dn}(v)$$

where $z = \exp(\pi \mathrm{i} v/I)$ and $\mathrm{cn}(v)$, $\mathrm{dn}(v)$ are Jacobi elliptic functions with half periods $I, \mathrm{i}I'$.

3.3. Integral formulae

In this section we calculate the trace of a product of two vertex operators and derive an integral formula of the probability function. The free field realizations of the degree operators are given by

$$D|_{V(\lambda)} = -\rho = -2\bar{d}^a - 2\bar{d}^\phi + \frac{1}{4}\partial^2_\alpha - \frac{1}{2}\partial_\alpha - \frac{(\lambda,\lambda)}{2} \qquad (\lambda = 2\Lambda_0, 2\Lambda_1, \Lambda_0 + \Lambda_1). \tag{20}$$

Here we set

$$\bar{d}^a = \sum_{m=0}^\infty m N^a_m \qquad \bar{d}^\phi = \sum_{k>0} k N^\phi_k$$

where

$$N^a_m = \frac{m}{[2m]^2} a_{-m}a_m \qquad N^\phi_k = \frac{1}{x^{2k} + x^{-2k}}\phi_{-k}\phi_k.$$

We divide the trace on $V(2\Lambda_0) \simeq F^{(0)}_+$ into three parts;

$$\begin{aligned} \mathrm{tr}_{V(2\Lambda_0)}(x^{2D}\ldots) &= \mathrm{tr}_{F^a}(x^{-4\bar{d}^a}\ldots)\cdot \mathrm{tr}_{F^{\phi NS}_+}(x^{-4d^{\phi NS}}\ldots)\cdot \mathrm{tr}_{F[2Q]}(x^{\frac{1}{2}\partial^2_\alpha - \partial_\alpha}\ldots) \\ &\quad + \mathrm{tr}_{F^a}(x^{-4\bar{d}^a}\ldots)\cdot \mathrm{tr}_{F^{\phi NS}_-}(x^{-4d^{\phi NS}}\ldots)\cdot \mathrm{tr}_{\mathrm{e}^\alpha F[2Q]}(x^{\frac{1}{2}\partial^2_\alpha - \partial_\alpha}\ldots). \end{aligned}$$

The fermion parts can be written as

$$\mathrm{tr}_{F^{\phi NS}_\pm}(x^{-4d^{\phi NS}}\ldots) = \tfrac{1}{2}(\mathrm{tr}_{F^{\phi NS}}(x^{-4d^{\phi NS}}\ldots) \pm \mathrm{tr}_{F^{\phi NS}}((\mathrm{i}x)^{-4d^{\phi NS}}\ldots)).$$

Now we consider the trace of a product of two vertex operators

$$\mathrm{tr}_{V(2\Lambda_0)}(x^{2D}\Phi_\epsilon(x^{-1}\zeta)\Phi_{1-\epsilon}(\zeta)) \qquad \text{for} \quad \epsilon = 0, 1.$$

The trace taken over bosonic space is a direct consequence of the following formulae:

$$\mathrm{tr}_{F^a}\left(y^{-2\bar{d}^a}\exp\left(\sum_{n=1}^{\infty}A_n a_{-n}\right)\exp\left(\sum_{n=1}^{\infty}B_n a_n\right)\right)$$
$$= \exp\left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}y^{2mn}A_nB_n\frac{[2n]^2}{n}\right)\prod_{m=1}^{\infty}\frac{1}{1-y^{2m}}$$

and

$$\left(x^6\left(\frac{\zeta_1}{\zeta_2}\right)^2; x^4, x^4\right)_\infty^2 B_{I,>}(\zeta_2)B_{I,<}(\zeta_1)$$
$$= \left(x^4\left(\frac{\zeta_1}{\zeta_2}\right)^2; x^4, x^4\right)_\infty\left(x^8\left(\frac{\zeta_1}{\zeta_2}\right)^2; x^4, x^4\right)_\infty B_{I,<}(\zeta_1)B_{I,>}(\zeta_2). \tag{21}$$

The traces taken over bosonic space F^a can be written as infinite products.

$$\mathrm{tr}_{F^a}(y^{-2\bar{d}^a}B_{I,<}(\zeta_2)B_{I,<}(\zeta_1)E_<^-(w)B_{I,>}(\zeta_2)B_{I,>}(\zeta_1)E_>^-(w))$$
$$= \frac{(x^4y^2; x^4, x^4, y^2)_\infty^2(x^8y^2; x^4, x^4, y^2)_\infty^2}{(x^6y^2; x^4, x^4, y^2)_\infty^4}\cdot\frac{(x^2y^2; y^2)_\infty}{(y^2; y^2)_\infty}$$
$$\times\left[\frac{(x^4y^2(\frac{\zeta_2}{\zeta_1})^2; x^4, x^4, y^2)_\infty(x^8y^2(\frac{\zeta_2}{\zeta_1})^2; x^4, x^4, y^2)_\infty}{(x^6y^2(\frac{\zeta_2}{\zeta_1})^2; x^4, x^4, y^2)_\infty^2}\right.$$
$$\left.\times\frac{(-x^9y^2\frac{\zeta_2^2}{w}; x^4, y^2)_\infty(-xy^2\frac{w}{\zeta_2^2}; x^4, y^2)_\infty}{(-x^7y^2\frac{\zeta_2^2}{w}; x^4, y^2)_\infty(-x^{-1}y^2\frac{w}{\zeta_2^2}; x^4, y^2)_\infty}\times(\zeta_2\leftrightarrow\zeta_1)\right]. \tag{22}$$

The traces taken over lattice space have the following theta function formulae:

$$\mathrm{tr}_{F[2Q]}(x^{\frac{1}{2}\partial_\alpha^2-\partial_\alpha}f^{\partial_\alpha}) = \sum_{l\in\mathbb{Z}}x^{8l^2-4l}f^{4l} = \Theta_{x^{16}}(-x^4f^4) \tag{23}$$

$$\mathrm{tr}_{e^\alpha F[2Q]}(x^{\frac{1}{2}\partial_\alpha^2-\partial_\alpha}f^{\partial_\alpha}) = \sum_{l\in\mathbb{Z}}x^{8l^2+4l}f^{4l+2} = f^2\Theta_{x^{16}}(-x^{12}f^4). \tag{24}$$

Here we use the standard notation of the theta function, defined as

$$\Theta_p(z) = (p; p)_\infty(z; p)_\infty(pz^{-1}; p)_\infty.$$

Now we concentrate on the trace taken over the fermionic space:

$$\mathrm{tr}_{F_\pm^{\phi NS}}(x^{-4\bar{d}^\phi}\Omega_R^{NS}(\zeta/x)\Omega_{NS}^R(\zeta)\phi^{NS}(w))$$
$$= \mathrm{tr}_{F_\pm^{\phi NS}}\left(x^{-4\bar{d}^\phi}\Phi_R^{NS,\pm}\left(-\frac{\mathrm{i}}{x^{\frac{1}{2}}\zeta}\right)\Phi_{NS,\mp}^R\left(-\frac{\mathrm{i}}{x^{\frac{3}{2}}\zeta}\right)\phi^{NS}(w)\right) \tag{25}$$

where the symbols $\Phi_{NS,\pm}^R(\zeta)$ and $\Phi_R^{NS,\pm}(\zeta)$ are type-I vertex operators of the two-dimensional Ising model. From relations (17) and (19), we deform the vertex operators in (25) to the fermion currents. Only the fermion currents and the degree operator appear in the trace:

$$\pm\frac{\mathrm{i}}{2}g^{NS}\left\{\mathrm{tr}_{F^{\phi NS}}\left(x^{-4\bar{d}^\phi}\psi_1^{NS}\left(-\frac{\mathrm{i}}{x^{\frac{3}{2}}\zeta}\right)\phi^{NS}(w)\right)\right.$$
$$\left.\pm\,\mathrm{tr}_{F^{\phi NS}}\left((\mathrm{i}x)^{-4\bar{d}^\phi}\psi_1^{NS}\left(-\frac{\mathrm{i}}{x^{\frac{3}{2}}\zeta}\right)\phi^{NS}(w)\right)\right\}. \tag{26}$$

Here we use

$$\psi_1^{NS}(\zeta) = \oint \frac{dz}{2\pi i z} f_0^{NS}(z)\phi^{NS}(z/\zeta^2) \qquad f_0^{NS}(z) = 2\sqrt{x}(x^4;x^4)_\infty(-x^4;x^4)_\infty^2 \mathrm{cn}(v).$$

To take the trace of (26) we invoke the following simple relation:

$$\frac{\mathrm{tr}_{F^{\phi^{NS}}}(\xi^{-2\bar{d}^\phi}\phi^{NS}(w_1)\phi^{NS}(w_2))}{\mathrm{tr}_{F^{\phi^{NS}}}(\xi^{-2\bar{d}^\phi})} = \sum_{m\in\mathbb{Z}+\frac{1}{2}} \frac{x^{2m}+x^{-2m}}{1+\xi^{2m}}\left(\frac{w_2}{w_1}\right)^m.$$

We calculate the trace taken over fermionic space and calculate the integrals in (26), using the Fourier expansion of coefficient function $f_0^{NS}(z)$ given by

$$f_0^{NS}(z) = \frac{1}{\sqrt{x}(x^4;x^4)_\infty(-x^4;x^4)_\infty^2}\sum_{r\in\mathbb{Z}+\frac{1}{2}}\frac{1}{x^{2r}+x^{-2r}}z^r.$$

We have the following infinite sum formulae:

$$\pm\frac{i}{2}g^{NS}\frac{1}{\sqrt{x}(x^4;x^4)_\infty(-x^4;x^4)_\infty^2}\left\{\mathrm{tr}_{F^{\phi^{NS}}}(x^{-4\bar{d}^\phi})\sum_{m\in\mathbb{Z}+\frac{1}{2}}\frac{1}{x^{2m}+x^{-2m}}\left(-\frac{w}{x^5\zeta^2}\right)^m \right.$$
$$\left.\mp\mathrm{tr}_{F^{\phi^{NS}}}((ix)^{-4\bar{d}^\phi})\sum_{m\in\mathbb{Z}+\frac{1}{2}}\frac{1}{x^{2m}-x^{-2m}}\left(-\frac{w}{x^5\zeta^2}\right)^m\right\}.$$

Using the following theta function's identities:

$$\sum_{m\in\mathbb{Z}+\frac{1}{2}}\frac{1}{x^{2m}\pm x^{-2m}}z^m = \pm x\sqrt{z}\frac{(x^4;x^4)_\infty^2}{(\mp x^2;x^4)_\infty^2}\frac{\Theta_{x^4}(\mp x^4 z)}{\Theta_{x^4}(x^2 z)}$$

we get the infinite product formula of the trace taken over fermionic space

$$\begin{aligned}
&\mathrm{tr}_{F_\pm^{\phi^{NS}}}(x^{-4\bar{d}^\phi}\Omega_R^{NS}(\zeta/x)\Omega_{NS}^R(\zeta)\phi^{NS}(w))\\
&\quad = \mp\frac{1}{2}\frac{w^{\frac{1}{2}}}{\zeta x^2}\frac{(x^4;x^4)_\infty^2}{(-x^4;x^4)_\infty^2}\frac{(x^8;x^4,x^8)_\infty^2}{(x^6;x^4,x^4)_\infty}\frac{1}{\Theta_{x^4}\left(-\frac{w}{x^3\zeta^2}\right)}\\
&\qquad\times\left\{(-x^2;x^4)_\infty\frac{\Theta_{x^4}\left(\frac{w}{x\zeta^2}\right)}{\Theta_{x^4}(-x^2)}\pm(x^2;x^4)_\infty\frac{\Theta_{x^4}(-\frac{w}{x\zeta^2})}{\Theta_{x^4}(x^2)}\right\}.
\end{aligned}\tag{27}$$

Here we use the character formula of fermion Fock space;

$$\mathrm{tr}_{F^{\phi^{NS}}}(x^{-4\bar{d}^\phi}) = (-x^2;x^4)_\infty.$$

Combining relations (21)–(24) and (27), we get an integral formula of the spontaneous staggered polarizations. We can summarize the conclusion obtained as follows.

The trace of a product of two vertex operators has following integral formulae:

$$\begin{aligned}
&\mathrm{tr}_{V(2\Lambda_0)}(x^{2D}\Phi_{\Lambda_0+\Lambda_1,\epsilon}^{2\Lambda_0\,*}(\zeta)\Phi_{2\Lambda_0,\epsilon}^{\Lambda_0+\Lambda_1}(\zeta))\\
&\quad=\frac{1}{2x^3\zeta^2}\frac{(x^8;x^4,x^8)_\infty^2(x^{10};x^4,x^4)_\infty}{(x^8;x^4,x^4)_\infty^2}\frac{(x^6;x^4)_\infty^2(x^4;x^4)_\infty^2}{(-x^4;x^4)_\infty^2}\\
&\qquad\times\oint_{C_{1+\epsilon}}\frac{dw}{2\pi i w}\{(1+x^2)w+2x^{5-2\epsilon}\zeta^2\}\frac{1}{\Theta_{x^4}(-\frac{w}{x\zeta^2})\Theta_{x^4}(-\frac{w}{x^3\zeta^2})}\\
&\qquad\times\left[\left\{\frac{\Theta_{x^4}(\frac{w}{x\zeta^2})}{\Theta_{x^4}(-x^2)}(-x^2;x^4)_\infty+\frac{\Theta_{x^4}(-\frac{w}{x\zeta^2})}{\Theta_{x^4}(x^2)}(x^2;x^4)_\infty\right\}\right.
\end{aligned}$$

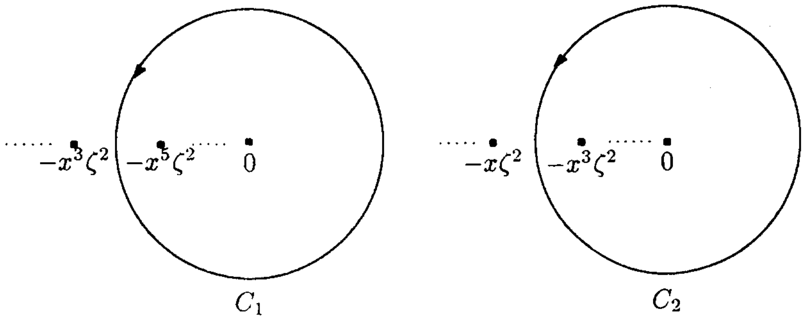

Figure 9. Contors.

$$\times\Theta_{x^{16}}\left(-x^6\left(\frac{w}{\zeta^2}\right)^2\right)$$
$$-\left\{\frac{\Theta_{x^4}(\frac{w}{x\zeta^2})}{\Theta_{x^4}(-x^2)}(-x^2;x^4)_\infty-\frac{\Theta_{x^4}(-\frac{w}{x\zeta^2})}{\Theta_{x^4}(x^2)}(x^2;x^4)_\infty\right\}$$
$$\times\, x^3\frac{\zeta^2}{w}\Theta_{x^{16}}\left(-\frac{1}{x^2}\left(\frac{w}{\zeta^2}\right)^2\right)\Bigg]. \tag{28}$$

Here the contours encircle $w=0$ in such a way that

$C_1: -x^5\zeta^2$ is inside and $-x^3\zeta^2$ is outside

$C_2: -x^3\zeta^2$ is inside and $-x\zeta^2$ is outside

as in figure 9.

3.4. Infinite product formula

The purpose of this section is to calculate the integral in (28) and derive an infinite product formula of spontaneous staggered polarization as in the main result.

Let us use the following abbreviations:

$$p_1(x)=\frac{(x^{16};x^{16})_\infty}{(x^4;x^4)^3_\infty}(-x^4;x^8)_\infty$$
$$p_2(x)=\frac{(x^{16};x^{16})_\infty}{(x^2;x^2)^2_\infty(x^4;x^4)_\infty}(-x^4;x^4)^2_\infty(-x^8;x^{16})^2_\infty$$
$$p_3(x)=\frac{(x^{16};x^{16})_\infty}{(x^2;x^2)^2_\infty(x^4;x^4)_\infty}(-x^4;x^4)^2_\infty(-x^{16};x^{16})^2_\infty$$
$$p_4(x)=\frac{(x^{16};x^{16})_\infty}{(x^2;x^2)^2_\infty(x^4;x^4)_\infty}(-x^2;x^4)^2_\infty(-x^4;x^8)_\infty.$$

Now we consider the following integral:

$$\oint\frac{\mathrm{d}w}{2\pi\mathrm{i}w}w\frac{\Theta_{x^4}(\frac{w}{x\zeta^2})\Theta_{x^{16}}(-x^6(\frac{w}{\zeta^2})^2)}{\Theta_{x^4}(-\frac{w}{x\zeta^2})\Theta_{x^4}(-\frac{w}{x^3\zeta^2})}.$$

The integrand function $I(z)=z\frac{\Theta_{x^4}(z)\Theta_{x^{16}}(-x^8z^2)}{\Theta_{x^4}(-z)\Theta_{x^4}(-z/x^2)}$ $(z=w/x\zeta^2)$ is an elliptic function and has odd invariance;

$$I(x^8z)=I(z)\qquad I(z)=-I(z^{-1}).$$

Therefore, taking the residue of Cauchy's principal value at $z = -1$, we have

$$\oint_{|z|=1-0} \frac{\mathrm{d}z}{2\pi \mathrm{i}z} I(z) = -\frac{1}{2}\mathrm{Res}_{z=-1} I(z).$$

Taking the residues near $w = -x\zeta^2$, we get the following formulae:

$$\oint_C \frac{\mathrm{d}w}{2\pi \mathrm{i}w} w \frac{\Theta_{x^4}(\frac{w}{x\zeta^2})\Theta_{x^{16}}(-x^6(\frac{w}{\zeta^2})^2)}{\Theta_{x^4}(-\frac{w}{x\zeta^2})\Theta_{x^4}(-\frac{w}{x^3\zeta^2})} = x^3\zeta^2 \times \begin{cases} p_2(x) - p_4(x) & C = C_1 \\ p_2(x) & C = C_2. \end{cases} \tag{29}$$

Using the same arguments as above, we get

$$\oint_C \frac{\mathrm{d}w}{2\pi \mathrm{i}w} \frac{\Theta_{x^4}(\frac{w}{x\zeta^2})\Theta_{x^{16}}(-\frac{1}{x^2}(\frac{w}{\zeta^2})^2)}{\Theta_{x^4}(-\frac{w}{x\zeta^2})\Theta_{x^4}(-\frac{w}{x^3\zeta^2})} = \begin{cases} -2x^2 p_3(x) + p_4(x) & C = C_1 \\ -2x^2 p_3(x) & C = C_2. \end{cases} \tag{30}$$

We consider the following integral:

$$\oint \frac{\mathrm{d}w}{2\pi \mathrm{i}w} w \frac{\Theta_{x^{16}}(-x^6(\frac{w}{\zeta^2})^2)}{\Theta_{x^4}(-\frac{w}{x^3\zeta^2})}.$$

The integrand function $J(z) = \frac{1}{z}\frac{\Theta_{x^{16}}(-z^2)}{\Theta_{x^4}(-x^2 z)}$ $(z = wx^3/\zeta^2)$ is an elliptic function and is composed of a product of two odd invariant functions $J_1(z) = \frac{\Theta_{x^{16}}(-z^2)}{\Theta_{x^{16}}(z^2)}$ and $J_2(z) = \frac{1}{z}\frac{\Theta_{x^{16}}(z^2)}{\Theta_{x^4}(-x^2 z)}$:

$$J(x^8 z) = J(z) \qquad J(z) = J_1(z)J_2(z)$$
$$J_k(x^8 z) = -J_k(z) \qquad J_k(z) = -J_k(z^{-1}) \qquad \text{for} \quad k = 1, 2.$$

From the odd invariance property, the constant term of Fourier expansion for variable u such that $z = \mathrm{e}^{\mathrm{i}u}$ becomes zero. Therefore, we have

$$\oint_{|z|=1} \frac{\mathrm{d}z}{2\pi \mathrm{i}z} J_1(z)J_2(z) = 0.$$

Taking the residue near $w = -x^3/\zeta^2$, we get the following formulae:

$$\oint_C \frac{\mathrm{d}w}{2\pi \mathrm{i}w} w \frac{\Theta_{x^{16}}(-x^6(\frac{w}{\zeta^2})^2)}{\Theta_{x^4}(-\frac{w}{x^3\zeta^2})} = \begin{cases} 0 & C = C_1 \\ x^3\zeta^2 p_1(x) & C = C_2. \end{cases} \tag{31}$$

Using the same arguments as above, we get the following formulae:

$$\oint_C \frac{\mathrm{d}w}{2\pi \mathrm{i}w} \frac{\Theta_{x^{16}}(-\frac{1}{x^2}(\frac{w}{\zeta^2})^2)}{\Theta_{x^4}(-\frac{w}{x^3\zeta^2})} = \begin{cases} p_1(x) & C = C_1 \\ 0 & C = C_2. \end{cases} \tag{32}$$

We consider the following integral:

$$\oint \frac{\mathrm{d}w}{2\pi \mathrm{i}w} \frac{\Theta_{x^{16}}(-x^6(\frac{w}{\zeta^2})^2)}{\Theta_{x^4}(-\frac{w}{x^3\zeta^2})}.$$

The integrand function $K(z) = \frac{\Theta_{x^{16}}(-z^2)}{\Theta_{x^4}(-z/x^6)}$, $(z = x^3 w/\zeta^2)$ satisfies the quasi-periodicity

$$K(z) = x^8 K(x^8 z).$$

Therefore, we have

$$\oint_{|z|=1} \frac{\mathrm{d}z}{2\pi \mathrm{i}z} K(z) = \frac{-1}{1-x^8}\left\{ \oint_{z=-x^2} + \oint_{z=-x^6} \right\} \frac{\mathrm{d}z}{2\pi \mathrm{i}z} K(z)$$

where we take the residues near ∞. Now we get the following formulae:

$$\oint_C \frac{\mathrm{d}w}{2\pi \mathrm{i} w} \frac{\Theta_{x^{16}}(-x^6(\frac{w}{\zeta^2})^2)}{\Theta_{x^4}(-\frac{w}{x^3\zeta^2})} = \begin{cases} \dfrac{1}{1+x^4} p_1(x) & C = C_1 \\ -\dfrac{x^4}{1+x^4} p_1(x) & C = C_2. \end{cases} \tag{33}$$

Using the same arguments as above we get the following formulae:

$$\oint_C \frac{\mathrm{d}w}{2\pi \mathrm{i} w} \frac{1}{w} \frac{\Theta_{x^{16}}(-\frac{1}{x^2}(\frac{w}{\zeta^2})^2)}{\Theta_{x^4}(-\frac{w}{x^3\zeta^2})} = \frac{1}{x^3\zeta^2} \begin{cases} -\dfrac{1}{1+x^4} p_1(x) & C = C_1 \\ \dfrac{x^4}{1+x^4} p_1(x) & C = C_2 \end{cases} \tag{34}$$

and

$$\begin{aligned} &\oint_C \frac{\mathrm{d}w}{2\pi \mathrm{i} w} \frac{1}{w} \frac{\Theta_{x^4}(\frac{w}{x\zeta^2})\Theta_{x^{16}}(-\frac{1}{x^2}(\frac{w}{\zeta^2})^2)}{\Theta_{x^4}(-\frac{w}{x\zeta^2})\Theta_{x^4}(-\frac{w}{x^3\zeta^2})} \\ &\quad = \frac{-1}{x\zeta^2(1-x^8)} \begin{cases} -2x^4 p_2(x) - 4x^2 p_3(x) + (x^2 + x^{-2}) p_4(x) & C = C_1 \\ -2x^4 p_2(x) - 4x^2 p_3(x) + (x^2 + x^6) p_4(x) & C = C_2 \end{cases} \end{aligned} \tag{35}$$

and

$$\begin{aligned} &\oint_C \frac{\mathrm{d}w}{2\pi \mathrm{i} w} \frac{\Theta_{x^4}(\frac{w}{x\zeta^2})\Theta_{x^{16}}(-x^6(\frac{w}{\zeta^2})^2)}{\Theta_{x^4}(-\frac{w}{x\zeta^2})\Theta_{x^4}(-\frac{w}{x^3\zeta^2})} \\ &\quad = \frac{-1}{1-x^8} \begin{cases} 2x^2 p_2(x) + 4x^8 p_3(x) - (1 + x^4) p_4(x) & C = C_1 \\ 2x^2 p_2(x) + 4x^8 p_3(x) - (x^4 + x^8) p_4(x) & C = C_2. \end{cases} \end{aligned} \tag{36}$$

Inserting relations (29)–(36) into integral formulae (28), we arrive at formulae (6) and (7). We have now proved the main result.

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References

[1] Jimbo M and Miwa T 1993 Algebraic analysis of solvable lattice models *CBMS Regional Conf. Ser. Math.* vol 85 (New York: AMS)
[2] Baxter R 1973 Spontaneous staggered polarization of F-model *J. Stat. Phys.* **9** 145–82
[3] Baxter R 1976 Corner transfer matrices of eight-vertex model: I *J. Stat. Phys.* **15** 485–503
[4] Davis B, Foda O, Jimbo M, Miwa T and Nakayashiki A 1993 Diagonalization of the XXZ Hamiltonian by vertex operators *Commun. Math. Phys.* **151** 89–153
[5] Lukyanov S and Pugai Y 1996 Multi-point local height probabilities in the integrable RSOS model *Nucl. Phys.* B **473** 631–58
[6] Idzumi M 1994 Level 2 irreducible representations of $U_q(\widehat{sl_2})$, vertex operators and their correlations *Int. J. Mod. Phys.* A **9** 4449–84
[7] Jimbo M, Kedem R, Kojima T, Konno H and and Miwa T 1995 XXZ-chain with a boundary *Nucl. Phys.* B **441** 437–70
[8] Lashkevich M and Pugai Y 1998 Free field construction for correlation functions of the eight-vertex model *Nucl. Phys.* B **516** 623–51

[9] Nakayashiki A 1992 Fusion of q-vertex operators and its application to solvable vertex models *Commun. Math. Phys.* **177** 27–62

[10] Fabrizio M and Gogolin A O 1994 Toulouse limit for the overscreened four-channel Kondo problem *Preprint* cond-mat/9407104

[11] Baxter R 1982 *Exactly Solved Models in Statistical Mechanics* (London: Academic)

[12] Idzumi M, Iohara K, Jimbo M, Miwa T, Nakashima T and Tokihiro T 1993 Quantum affine symmetry in vertex model *Int. J. Mod. Phys.* A **8** 1479–511

[13] Hara Y 1998 Free-field realization of vertex operators for level two modules of $U_q(\widehat{sl_2})$ *J. Phys. A: Math. Gen.* **31** 8483–94

[14] Foda O, Jimbo M, Miwa T, Miki K and Nakayashiki A 1994 Vertex operators in solvable lattice models *J. Math. Phys.* **35** 13–46